Research Statement

My research is focused in the area of algebra, specifically nilpotent Leibniz algebras of finite dimension. Due to the huge number of nilpotent Leibniz algebras, an uncountably infinite number for each dimension greater than 10, classifying them is done by placing further conditions on them. For my graduate research, I chose to restrict the algebras to those with isomorphic maximal subalgebras and classified them by coclass, which I define below. In order to discuss this, some background information is first necessary.

A Leibniz algebra A is a vector space over a field $\mathbb{F}$, equipped with a bilinear map $[\cdot,\cdot]: A \times A \to A$, called a multiplication, which satisfies the Leibniz identity

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$$

for all $a,b,c \in A$. The center of A is defined by $Z(A) = \{z \in A | [x,z] = 0 = [z,x] \, \forall x \in A\}$. The upper central series of A is given by

$$\{0\} = Z_0(A) \subset Z_1(A) \subset \cdots \subset Z_{c-1}(A) \subset Z_c(A) = A$$

where $Z(A) = Z_1(A)$ and $[Z_i(A), A] \subseteq Z_{i-1}(A)$ and $[A, Z_i(A)] \subseteq Z_{i-1}(A)$ for any $i \leq c$. Alternatively, $Z_{i+1}(A)/Z_i(A) = Z(A/Z_i(A))$. The class of A is c, and the coclass is given by $cc(A) = dim(A) - c$.

The key to thinking about coclass is understanding how the dimensions of $Z_i(A)$ relate to one another for different i. Since A is nilpotent, the dimension of $Z_i(A)$ must increase by at least one at every step. When the dimension of two consecutive terms increases by more than one, the result is a nonzero coclass. For example, if the coclass of A is one, then one of the terms of the upper central series increases in dimension by two instead of one. For coclass two, there must be one jump of dimension three, or two jumps of dimension two.

My research determined where these jumps in dimension could occur, which in turn placed restrictions on the structure of A and its upper central series. I was also able to gain information about certain quotient groups, such as $A/Z_2(A)$ or $A/Leib(A)$, where $Leib(A) = span\{[a,a] | a \in A\}$. My approach to classifying these algebras was to combine the knowledge of where jumps occur and the quotient groups to extract information about A. My work has classified nilpotent Leibniz algebras with isomorphic maximal subalgebras of coclass zero, one, and two over $\mathbb{C}$.

For further work, I will investigate classification problems using other restrictions, as well as explore related topics. A current subject in the fields of Lie and Leibniz algebras is the Schur multiplier, which is the kernel of the largest possible central extension of the algebra. Classifications for nilpotent algebras can be done based on how far the algebra's Schur multiplier is from being of maximal possible dimension. Other work continues to be done on Lie and Leibniz algebras of coclass one, often referred to as filiform. This work uses restrictions other than isomorphic maximal subalgebras, such as algebras which are positively graded or have short derived length. I will classify Leibniz algebras based on these restrictions. Unrelated to classification, Isoclinic extensions, which are built from isomorphisms and are related to Schur multipliers, provide another avenue of exploration. My work will determine under which conditions these results hold for Leibniz algebras.

Additionally, I plan to explore topics in post-Lie algebras, which is a Lie algebra with additional restrictions. Of particular interest to me are post-Lie algebra structures of matrices. Post-Lie algebra structures have been determined for $\mathfrak{gl}(2,\mathbb{C})$ and $\mathfrak{t}(2,\mathbb{C})$, which are the space of all 2×2 matrices over $\mathbb{C}$ and the space of all 2×2 upper triangular matrices over $\mathbb{C}$, respectively. I would expand these results to consider $\mathfrak{sl}(2,\mathbb{C})$, the space of 2×2 matrices over $\mathbb{C}$ with trace zero, and $\mathfrak{o}(2,\mathbb{C})$, the space of 2×2 skew-symmetric matrices over $\mathbb{C}$.

I greatly look forward to working with students in the future on a variety of research projects. Many of these projects stem from applications of algebra, as I have been teaching that subject for a couple of semesters. The first is having students work on the game Lights Out, in which the object is to turn off all lights on a 5×5 grid. However, toggling one light changes the adjacent lights as well. Having students extend this work to a larger grid and work examples of their solution with different starting states would make a good project.

Another possible project would extend the concept of "Mental Poker," where two people play 5-Card Draw without being in the same room. A cryptographic system is used to exchange cards without the players knowing each other's hands, or being able to cheat. Students could explore the use of different encryption systems and extending the game to Texas Hold'em, Go Fish, or Yahtzee. For dice games, the student would need to consider the challenge of how to keep track of dice order and groupings.

One other project would have students work on zero-knowledge proofs. In these proofs, one person proves their knowledge of a fact without revealing any other information. A process is established in which a person without knowledge of the fact would have a 50% chance of guessing correctly. Repeated successes will demonstrate knowledge of the fact. There are a broad range of subjects to which this concept can be applied, and students can come up with their own applications as well.

Alternatively, students looking to work more in pure mathematics would be able to participate in my Lie algebra research. Many of the calculations required for the post-Lie algebra structures given above would be accessible to students. Additionally, students could work on expanding results to 3×3 matrices as well.